Rank-Approximate Nearest Neighbor Search: Retaining Meaning and Speed in High Dimensions

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Abstract

The long-standing problem of efficient nearest-neighbor (NN) search has ubiquitous applications ranging from astrophysics to MP3 fingerprinting to bioinformatics to movie recommendations. As the dimensionality of the dataset increases, exact NN search becomes computationally prohibitive; $(1+\epsilon)$ distance-approximate NN search can provide large speedups but risks losing the meaning of NN search present in the ranks (ordering) of the distances. This paper presents a simple, practical algorithm allowing the user to, for the first time, directly control the true accuracy of NN search (in terms of ranks) while still achieving the large speedups over exact NN. Experiments with high-dimensional datasets show that it often achieves faster and more accurate results than the best-known distance-approximate method, with much more stable behavior.

1 Introduction

In this paper, we address the problem of nearest-neighbor (NN) search in large datasets of high dimensionality. It is used for classification (k-NN classifier [1]), categorizing a test point on the basis of the classes in its close neighborhood. Non-parametric density estimation uses NN algorithms when the bandwidth at any point depends on the k^{th} NN distance (NN kernel density estimation [2]). NN algorithms are present in and often the main cost of most non-linear dimensionality reduction techniques (manifold learning [3, 4]) to obtain the neighborhood of every point which is sought to be preserved during the dimension reduction. NN search has extensive applications in databases for answering queries by finding instances most similar to it in the database [5]. It is also used in computer vision for image search given a similarity measure between images. Further applications abound in machine learning.

Tree data structures such as kd-trees are used for efficient exact NN search but do not scale better than the naïve linear search in sufficiently high dimensions. Distance-approximate NN (DANN) search, introduced to increase the scalability of NN search, approximates the distance to the NN and any neighbor found within that distance is considered to be "good enough". Numerous techniques exist to achieve this form of approximation and are fairly scalable to higher dimensions under certain assumptions.

Although the DANN search places bounds on the numerical values of the distance to NN, in NN search, distances themselves are not essential; rather the order of the distances of the query to the points in the dataset captures the necessary and sufficient information [6, 7]. The proposed framework, *rank-approximate nearest-neighbor* (RANN) search, approximates the NN in its rank rather than in its distance. This model has the following advantage: a level of relative error in numerical distance in DANN becomes worse with increasing dimensions since the distribution of distances between points tends towards a small range [8]; however a particular relative error in the rank of the distances means the same thing in all dimensions.

This paper is organized as follows: Section 2 describes the existing methods for exact and distance-approximate NN search and the challenges they face in high dimensions. Section 3 introduces the proposed approach and provides a practical algorithm using stratified sampling with a tree data structure to obtain a user-specified level of rank approximation in Euclidean NN search. Section 4 reports the experiments comparing RANN with exact search and DANN. Finally, Section 5 concludes with discussion of the road ahead.

2 Related Work

The problem of NN search is formalized:

Problem. Given a dataset $S \subset X$ of size N in a metric space (X, d) and a query $q \in X$, efficiently find a point $p \in S$ such that

$$d(p, q) = \min_{r \in S} d(r, q). \tag{1}$$

2.1 Exact Search

The simplest approach of *linear search* over S to find the NN is easy to implement, but requires $\mathbf{O}(N)$ computations for a single NN query, making it unscalable for moderately large N.

Hashing the dataset into buckets is an efficient technique, but scales only to very low dimensional X. Hence data structures are used to answer queries efficiently. Binary spatial partitioning trees, like kd-trees [9], ball trees [10] and metric trees [11] utilize the triangular inequality of the distance metric d (commonly the Euclidean distance metric) to *prune* away parts of the dataset from the computation and answer queries in expected $\mathbf{O}(\log N)$ computations [9]. Non-binary cover trees [12] answer queries in theoretically bounded $\mathbf{O}(\log N)$ time using the same property under certain mild assumptions on the dataset.

Finding NNs for $\mathbf{O}(N)$ queries would then require at least $\mathbf{O}(N \log N)$ computations using the trees. The dual-tree algorithm [13] for NN search also builds a tree on the queries instead of going through them linearly, hence amortizing the cost of search over the queries. This algorithm shows orders of magnitude improvement in efficiency and is conjectured to be $\mathbf{O}(N)$ for answering $\mathbf{O}(N)$ queries using the cover trees [12].

2.2 Nearest Neighbors in High Dimensions

The frontier of research in NN methods is high dimensional problems, stemming from common datasets like images and documents to microarray data. But high dimensional data poses an inherent problem for Euclidean NN search as described in the following theorem:

Theorem 2.1. *[8] Let C be a $\mathcal{D}$-dimensional hypersphere with radius a. Let A and B be any two points chosen at random in C, the distributions of A and B being independent and uniform over the interior of C. Let r be the Euclidean distance between A and B ($r \in [0, 2a]$). Then the asymptotic distribution of r is $N(a\sqrt{2}, a^2/2\mathcal{D})$.*

This implies that as $\mathcal{D}$ goes up, the Euclidean distances between points approach a small range of continuous values. This renders the tree based algorithms no better than linear search since these data structures are unable to employ sufficiently tight bounds in high dimensions [14, 15, 16]. This prompted interest in approximation of the NN search problem.

2.3 Distance-Approximate Nearest Neighbors

The problem of NN search is relaxed in the following form to make it more scalable:

Problem. Given a dataset $S \subset X$ of size N in some metric space (X, d) and a query $q \in X$, efficiently find any point $p' \in S$ such that

$$d(p', q) \leq (1 + \epsilon) \min_{r \in S} d(r, q) \tag{2}$$

for a low value of $\epsilon \in \mathbb{R}^+$ with high probability.

This approximation can be achieved with kd-trees, balls trees, and cover trees by modifying the search algorithm to prune more aggressively. This introduces the allowed error while providing some speedup over the exact algorithm [12]. Another approach modifies the tree data structures to bound error with just one root-to-leaf traversal of the tree, i.e. to eliminate *backtracking*. Sibling nodes in kd-trees or ball-trees are modified to share points near their boundaries, forming *spill trees* [14]. These obtain significant speed up over the exact methods. The idea of *approximately correct* (satisfying Eq. 2) NN is further extended to a formulation where the $(1 + \epsilon)$ bound can be exceeded with a low probability δ, thus forming the PAC-NN search algorithms [17]. They provide 1-2 orders of magnitude speedup in moderately large datasets with suitable ϵ and δ.

All these methods are still unable to scale to high dimensionalities, but they may still be applied in combination with the assumption that high dimensional data actually lies on a lower dimensional subspace. There are a number of fast DANN methods that preprocess data with *randomized projections* to reduce dimensionality. *hybrid spill trees* [14] build *spill trees* on the randomly projected data to obtain significant speedups. *locality sensitive hashing* [18, 19] hashes the data into a lower dimensional buckets using hash functions which guarantee that "close" points are hashed into the same bucket with high probability and "farther apart" points are hashed into the same bucket with low probability. This method has significant improvements in running times over traditional methods in high dimensional data and is shown to be highly scalable.

However, the DANN methods lose the meaning present in the ordering of the NN distances [6]. Theorem 2.1 suggests that any Euclidean distance approximation $\epsilon \geq 1/(\sqrt{\mathcal{D}} - 0.5)$ makes no difference between the nearest and farthest neighbor with high probability. Thus in high dimensions, nearly any point satisfies the $(1 + \epsilon)$ approximation guarantee. So distance-approximation is meaningless in the absence of a true lower-dimensional subspace containing the data. In order to address these issues, we propose a model of approximation for NN search which preserves the information present in the ordering of the distances irrespective of the dimensionality of the dataset. We also provide a scalable algorithm to obtain this form of approximation.

3 Rank Approximation

To approximate the NN rank, we formulate and relax NN search in the following way:

Problem. Given a dataset $S \subset X$ of size N in a metric space (X, d) and a query $q \in X$, let $D = \{D_1, \ldots, D_N\}$ be the set of distances between the query and all the points in the dataset S, such that $D_i = d(q, r_i), r_i \in S, i = 1, \ldots, N$. Let $D_{(r)}$ be the r^{th} order statistic of D. Then the $r \in S$: $d(q, r) = D_{(1)}$ is the NN of q in S. The rank-approximation of NN search would then be to efficiently find a point $p' \in S$ such that

$$d(q, p') \leq D_{(1+\tau)} \tag{3}$$

with high probability for a given value of $\tau \in \mathbb{Z}^+$.

RANN search may use any order statistics of the population D, bounded above by the $(1 + \tau)^{th}$ order statistics. Sedransk et.al. [20] provide a probability we can use to bound the order statistics of a set given the order statistics of a sampled subset:

Theorem 3.1. *For a population consisting of N elements with Y values ordered as $Y_{(1)} \leq Y_{(2)} \cdots \leq Y_{(N)}$, let $y_{(1)} \leq y_{(2)} \cdots \leq y_{(n)}$ be a sample of size n drawn from the population uniformly without replacement. For $1 \leq t \leq N$ and $1 \leq k \leq n$,*

$$P(y_{(k)} \leq Y_{(t)}) = \sum_{i=0}^{t-k} \binom{t-i-1}{k-1} \binom{N-t+i}{n-k} / \binom{N}{n}. \tag{4}$$

We may find a $p' \in S$ satisfying Eq. 3 with high probability by sampling enough points $\{d_1, \ldots d_n\}$ from D such that for some $1 \leq k \leq n$, rank error bound τ, and a success probability α

$$P(d_{(k)} \leq D_{(1+\tau)}) \geq \alpha. \tag{5}$$

Sample order statistic $k = 1$ minimizes the needed number of samples; hence we substitute the values of $k = 1$ and $t = 1 + \tau$ in Eq. 4 obtaining the following expression which can be computed in $\mathbf{O}(\tau)$ time

$$P(d_{(1)} \leq D_{(1+\tau)}) = \sum_{i=0}^{\tau} \left(\begin{array}{c} N-\tau+i-1 \\ n-1 \end{array} \right) / \left(\begin{array}{c} N \\ n \end{array} \right). \quad (6)$$

The required sample size n for a particular error τ with success probability α is computed using binary search over the range $(1 + \tau, N]$. This makes RANN search $\mathbf{O}(n)$ (since now we only need to compute the first order statistics of a sample of size n) giving $\mathbf{O}(N/n)$ speedup.

3.1 Stratified Sampling with a Tree

When the required number of samples is n, it means that we require to randomly sample n points from S, compute the distance of the query q from each of these points and subsequently obtain the first order sample statistics. But if the dataset S is in the form of a tree, parts of the dataset can be pruned away for q and hence, the distance of q to points sampled from the pruned part of the dataset is no longer required to be computed. Thus, the number of points n' for which the distance to q must be computed is less than n.

Hence let S be in the form of a binary tree (say kd-tree) rooted at R_{root}. The root node has N points. Let the left and right child have N_l and N_r points respectively. For a random query $q \in X$, the population D is the set of distances of q to all the N points in R_{root}. The tree stratifies the population D into $D_l = \{D_{l1}, \ldots, D_{lN_l}\}$ and $D_r = \{D_{r1}, \ldots, D_{rN_r}\}$, where D_l and D_r are the set of distances of q to all the N_l and N_r points respectively in the left and right child of the root node R_{root}. The following theorem provides a way to decide how much to sample from a particular node, subsequently providing a lower bound on the number of samples required from the unpruned part of the tree without violating Eq.5

Theorem 3.2. *Let n_l and n_r be the number of random samples from the strata D_l and D_r respectively by doing a stratified sampling on the population D of size $N = N_l + N_r$. Let n samples be required for Eq.5 to hold in the population D for a given value of α. Then Eq.5 holds for D with the same value of α with the random samples of sizes n_l and n_r from the random strata D_l and D_r of D respectively if $n_l + n_r = n$ and $n_l : n_r = N_l : N_r$.*

Proof. Eq. 5 simply requires n uniformly sampled points, i.e. for each distance in D to have probability n/N of inclusion. For $n_l + n_r = n$ and $n_l : n_r = N_l : N_r$, we have $n_l = \lceil (n/N)N_l \rceil$ and similarly $n_r = \lceil (n/N)N_r \rceil$, and thus samples in both D_l and D_r are included at the proper rate. □

Since the ratio of the sample size to the population size is a constant $\beta = n/N$, Theorem 3.2 is generalizable to any level of the tree.

3.2 The Algorithm

The intended approximation is introduced in the unpruned part of the tree since the pruned part does not add to the computation in the exact tree based algorithms. While searching for the NN of a query q in a tree, most of the computation in the traversal is obtaining the distance $dist\_to\_node(q, R)$ of a query to any tree node R. If the current upperbound to the NN distance $ub(q)$ is greater than $dist\_to\_node(q, R)$, the node is traversed. Otherwise it is pruned away. The computations of distance of q to points in the dataset S occurs only when the query reaches a leaf node it cannot prune. The NN candidate in that leaf is computed using linear search (COMPUTEBRUTENN subroutine in Fig.2). The traversal of the exact algorithm in the tree is illustrated in Fig.1.

To approximate the computation by sampling, traversal to the children of a node in the tree is stopped at a level when the number of samples required from that node to obtain the required level of approximation goes below a certain threshold MAXSAMPLES. This is illustrated in Fig.1. The value of MAXSAMPLES giving maximum speedup can be obtained by cross-validation. If a node is summarizable within the desired error bounds (decided by the CANAPPROXIMATE subroutine in Fig.2), required number of points are sampled from such a node and the nearest neighbor candidate is computed from among them using linear search (COMPUTEAPPROXNN subroutine of Fig.2).

Single Tree. The search algorithm is presented in Fig.2. The dataset S is stored as a binary tree rooted at R_{root}. The algorithm starts as STRANKAPPROXNN(q, S, τ, α). During the search, if a

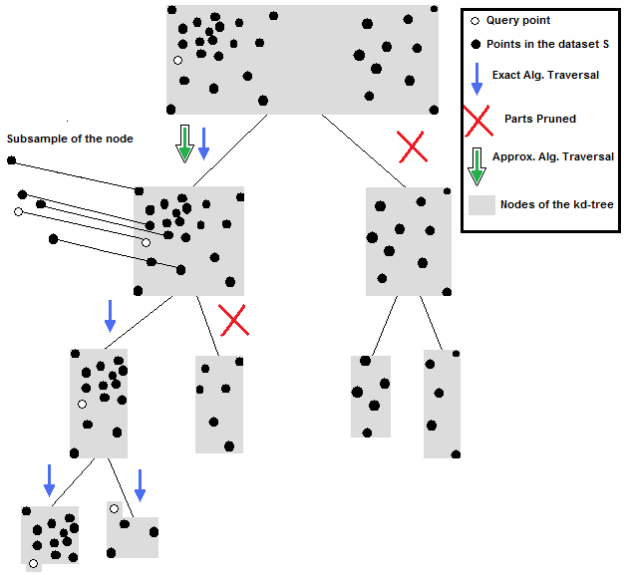


Figure 1: The traversal paths of the exact and the rank-approximate algorithm in a kd-tree

leaf node is reached (since the tree is rarely balanced), the exact NN candidate is computed. In case a non-leaf node cannot be approximated, the child node closer to the query is always traversed first. The following theorem proves the correctness of the algorithm.

Theorem 3.3. *For a query q and a specified value of α and τ,* STRANKAPPROXNN(q, S, τ, α) *computes a neighbor in S within $(1+\tau)$ rank with probability at least α.*

Proof. By Eq.6, a query requires at least n samples from a dataset of size N to compute a neighbor within $(1+\tau)$ rank with a probability α. Let $\beta = (n/N)$. Let a node R contain $|R|$ points. In the algorithm, sampling occurs when a base case of the recursion is reached. There are three base cases:

- Case 1 - Exact Pruning (**if** $ub(q) \leq dist\_to\_node(q, R)$): Then number of points required to be sampled from the node is at least $\lceil \beta \cdot |R| \rceil$. However, since this node is pruned, we do not need to compute the distances from the query to these points. Hence nothing is done in the algorithm.
- Case 2 - Exact Computation COMPUTEBRUTENN(q, R)): In this subroutine, linear search is used to find the NN candidate. Hence number of points actually sampled is $|R| \geq \lceil \beta \cdot |R| \rceil$.
- Case 3 - Approximate Computation (COMPUTEBRUTENN(q, R, β)): In this subroutine, exactly $\beta \cdot |R|$ samples are made and linear search is performed over them.

Let the total number of points sampled from S be n'. From the three base cases of the algorithm, it is confirmed that $n' \geq \lceil \beta \cdot N \rceil = n$. Hence the algorithm computes a NN within $(1+\tau)$ rank with probability at least α. □

Dual Tree. The single tree algorithm in Fig.2 can be modified to use the dual tree concept in case of $\mathbf{O}(N)$ queries. The dual tree RANN algorithm (DTRANKAPPROXNN(T, S, τ, α)) is given in Fig.2. The only difference is that for every query $q \in T$, the minimum required amount of sampling is done and the random sampling is done separately for each of the queries. Even though the queries do not share samples from the reference set, when a query node of the query tree prunes a reference node, that reference node is pruned for all the queries in that query node simultaneously. This work-sharing is a key feature of all dual-tree algorithms [13].

STRANKAPPROXNN(q, S, τ, α)

$n \leftarrow$COMPUTESAMPLESIZE ($|S|, \tau, \alpha$)
$\beta \leftarrow n/|S|$
$R_{root} \leftarrow$TREE(S)
STRANN(q, R_{root}, β)

STRANN(q, R, β)

if $ub(q) > dist\_to\_node(q, R)$ **then**
  if ISLEAF(R) **then**
    COMPUTEBRUTENN(q, R)
  else if CANAPPROXIMATE(R, β) **then**
    COMPUTEAPPROXNN (q, R, β)
  else
    STRANN(q, R^l, β),
    STRANN(q, R^r, β)
  end if
end if

COMPUTEBRUTENN(q, R)

$ub(q) \leftarrow \min(\min_{r \in R} d(q, r), ub(q))$

COMPUTEBRUTENN(Q, R)

for $\forall q \in Q$ **do**
  $ub(q) \leftarrow \min(\min_{r \in R} d(q, r), ub(q))$
end for
$node\_ub(Q) \leftarrow \max_{q \in Q} ub(q)$

COMPUTEAPPROXNN(q, R, β)

$R' \leftarrow \lceil \beta \cdot |R| \rceil$ samples from R
COMPUTEBRUTENN(q, R')

COMPUTEAPPROXNN(Q, R, β)

for $\forall q \in Q$ **do**
  $R' \leftarrow \lceil \beta \cdot |R| \rceil$ samples from R
  COMPUTEBRUTENN(q, R')
end for
$node\_ub(Q) \leftarrow \max_{q \in Q} ub(q)$

DTRANKAPPROXNN(T, S, τ, α)

$n \leftarrow$COMPUTESAMPLESIZE ($|S|, \tau, \alpha$)
$\beta \leftarrow n/|S|$
$R_{root} \leftarrow$TREE(S)
$Q_{root} \leftarrow$TREE(T)
DTRANN($Q_{root}, R_{root}, \beta$)

DTRANN(Q, R, β)

if $node\_ub(Q) > dist\_between\_nodes(Q, R)$ **then**
  if ISLEAF(Q) && ISLEAF(R) **then**
    COMPUTEBRUTENN(Q, R)
  else if ISLEAF(R) **then**
    DTRANN(Q^l, R, β), DTRANN(Q^r, R, β)
    $node\_ub(Q) \leftarrow \max_{i=\{l,r\}} node\_ub(Q^i)$
  else if CANAPPROXIMATE(R, β) **then**
    if ISLEAF(Q) **then**
      COMPUTEAPPROXNN (Q, R, β)
    else
      DTRANN(Q^l, R, β),
      DTRANN(Q^r, R, β)
      $node\_ub(Q) \leftarrow \max_{i=\{l,r\}} node\_ub(Q^i)$
    end if
  else if ISLEAF(Q) **then**
    DTRANN(Q, R^l, β), DTRANN(Q, R^r, β)
  else
    DTRANN(Q^l, R^l, β), DTRANN(Q^l, R^r, β)
    DTRANN(Q^r, R^l, β),
    DTRANN(Q^r, R^r, β)
    $node\_ub(Q) \leftarrow \max_{i=\{l,r\}} node\_ub(Q^i)$
  end if
end if

CANAPPROXIMATE(R, β)

return $\lceil \beta \cdot |R| \rceil \leq$MAXSAMPLES

Figure 2: Single tree (STRANKAPPROXNN) and dual tree (DTRANKAPPROXNN) algorithms and subroutines for RANN search for a query q (or a query set T) in a dataset S with rank approximation ϵ and success probability α. R^l and R^r are the closer and farther child respectively of R from the query q (or a query node Q)

4 Experiments and Results

A meaningful value for the rank error τ should be relative to the size of the reference dataset N. Hence for the experiments, the $(1 + \tau)$-RANN is slightly modified to $(1 + \lceil \varepsilon \cdot N \rceil)$-RANN where $1.0 \geq \varepsilon \in \mathbb{R}^+$. The Euclidean metric is used in all the experiments. Although the value of MAXSAMPLES for maximum speedup can be obtained by cross-validation, for practical purposes, any low value ($\approx$ 20-30) suffices well, and this is what is used in the experiments.

4.1 Comparisons with Exact Search

The speedups of the exact dual-tree NN algorithm and the approximate tree-based algorithm over the linear search algorithm is computed and compared. Different levels of approximations ranging from 0.001% to 10% are used to show how the speedup increases with increase in approximation.

Different datasets drawn for the UCI repository (Bio dataset 300k$\times$74, Corel dataset 40k$\times$32, Covertype dataset 600k$\times$55, Phy dataset 150k$\times$78)[21], MNIST handwritten digit recognition

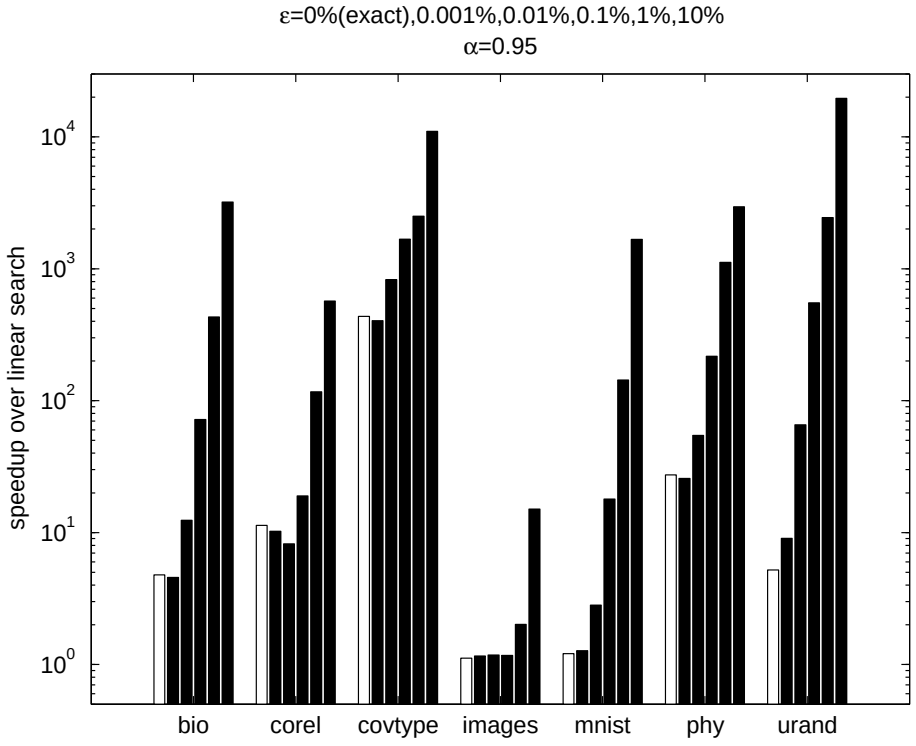


Figure 3: Speedups(logscale on the Y-axis) over the linear search algorithm while finding the NN in the exact case or $(1 + \varepsilon N)$-RANN in the approximate case with $\varepsilon = 0.001\%, 0.01\%, 0.1\%, 1.0\%, 10.0\%$ and a fixed success probability $\alpha = 0.95$ for every point in the dataset. The first(white) bar in each dataset in the X-axis is the speedup of exact dual tree NN algorithm, and the subsequent(dark) bars are the speedups of the approximate algorithm with increasing approximation.

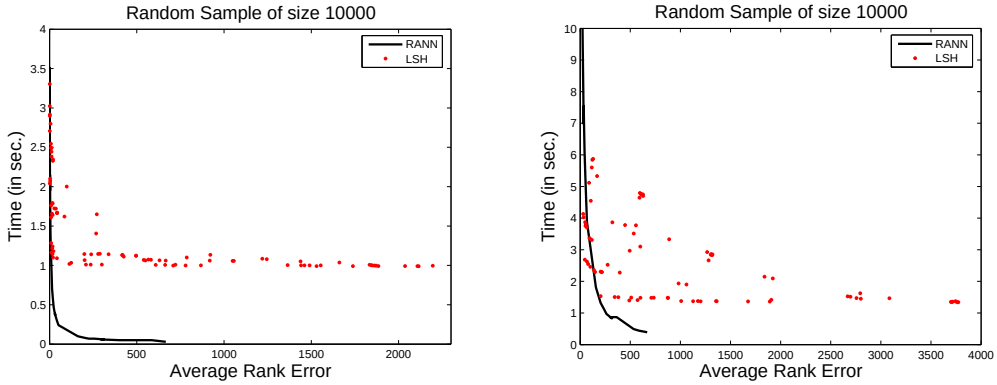
dataset (60k$\times$784)[22] and the Isomap "images" dataset (700$\times$4096)[3] are used. The final dataset "urand" is a synthetic dataset of points uniform randomly sampled from a unit ball (1m$\times$20). This dataset is used to show that even in the absence of a lower-dimensional subspace, RANN is able to get significant speedups over exact methods for relatively low errors. For each dataset, the NN of every point in the dataset is found in the exact case, and $(1+\lceil \varepsilon \cdot N \rceil)$-rank-approximate NN of every point in the dataset is found in the approximate case. These results are summarized in Fig.3.

The results show that for even low values of ε (high accuracy setting), the RANN algorithm is significantly more scalable than the exact algorithms for all the datasets. Note that for some of the datasets, the low values of approximation used in the experiments are equivalent to zero rank error (which is the exact case), hence are equally efficient as the exact algorithm.

4.2 Comparison with Distance-Approximate Search

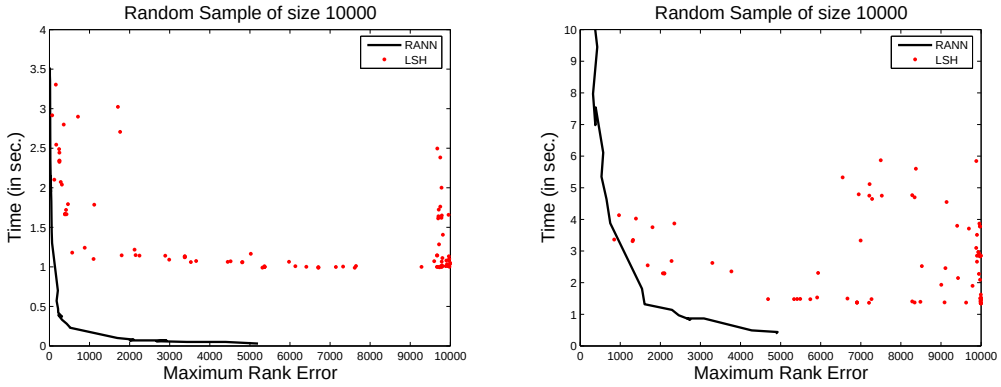
In the case of the different forms of approximation, the average rank errors and the maximum rank errors achieved in comparable retrieval times are considered for comparison. For DANN, Locality Sensitive Hashing (LSH) [19, 18] is used.

Subsets of two datasets known to have a lower-dimensional embedding are used for this experiment - Layout Histogram (10k$\times$30)[21] and MNIST dataset (10k$\times$784)[22]. The approximate NN of every point in the dataset is found with different levels of approximation for both the algorithms. The average rank error and maximum rank error is computed for each of the approximation levels. For our algorithm, we increased the rank error and observed a corresponding decrease in the retrieval time. LSH has three parameters. To obtain the best retrieval times with low rank error, we fixed one



(a) Layout Histogram (b) Mnist

Figure 4: Query times on the X-axis and the Average Rank Error on the Y-axis.



(a) Layout Histogram (b) Mnist

Figure 5: Query times on the X-axis and the Maximum Rank Error on the Y-axis.

parameter and changed the other two to obtain a decrease in runtime and did this for many values of the first parameter. The results are summarized in Fig. 4 and Fig. 5.

The results show that even in the presence of a lower-dimensional embedding of the data, the rank errors for a given retrieval time are comparable in both the approximate algorithms. The advantage of the rank-approximate algorithm is that the rank error can be directly controlled, whereas in LSH, tweaking in the cross-product of its three parameters is typically required to obtain the best ranks for a particular retrieval time. Another advantage of the tree-based algorithm for RANN is the fact that even though the maximum error is bounded only with a probability, the actual maximum error is not much worse than the allowed maximum rank error since a tree is used. In the case of LSH, at times, the actual maximum rank error is extremely large, corresponding to LSH returning points which are very far from being the NN. This makes the proposed algorithm for RANN much more stable than LSH for Euclidean NN search. Of course, the reported times highly depend on implementation details and optimization tricks, and should be considered carefully.

5 Conclusion

We have proposed a new form of approximate algorithm for unscalable NN search instances by controlling the true error of NN search (i.e. the ranks). This allows approximate NN search to retain meaning in high dimensional datasets even in the absence of a lower-dimensional embedding. The proposed algorithm for approximate Euclidean NN has been shown to scale much better than the exact algorithm even for low levels of approximation even when the true dimension of the data is

relatively high. When compared with the popular DANN method (LSH), it is shown to be comparably efficient in terms of the average rank error even in the presence of a lower dimensional subspace of the data (a fact which is crucial for the performance of the distance-approximate method). Moreover, the use of spatial-partitioning tree in the algorithm provides stability to the method by clamping the actual maximum error to be within a reasonable rank threshold unlike the distance-approximate method.

However, note that the proposed algorithm still benefits from the ability of the underlying tree data structure to bound distances. Therefore, our method is still not necessarily immune to the curse of dimensionality. Regardless, RANN provides a new paradigm for NN search which is comparably efficient to the existing methods of distance-approximation and allows the user to directly control the true accuracy which is present in ordering of the neighbors.

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